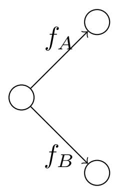
An Extension of IRR for Random Variables

Consider a set of risky assets held in a portfolio according to predetermined fractions where assets yield a return on a fixed horizon. When any of the assets reaches maturity, the money coming back (represented by a random variable) will be reinvested into a portfolio of the same constituency.

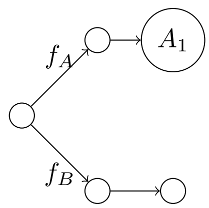
We model this scenario as a branching process defined by the series X_n, visualized below.

$$X_0 = 1 =$$

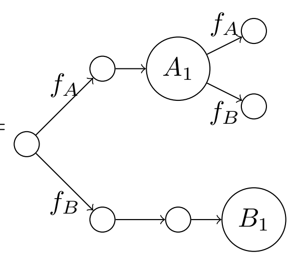
$$X_1 = f_A + f_B =$$



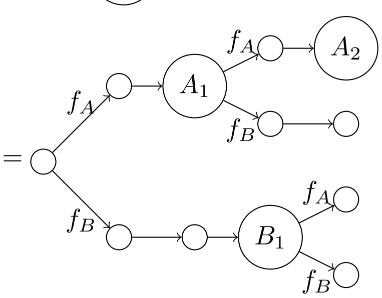
$$X_2 = f_A A_1 + f_B =$$



$$X_3 = f_A A_1 (f_A + f_B) + f_B B_1 =$$



$$X_3 = f_A A_1 (f_A A_2 + f_B) + f_B B_1 (f_A + f_B) =$$



$$\vdots$$

We define IRR in this context to be the long-term growth rate of the series

$$I_a = \lim_{N \to \infty} \sqrt[N]{X_N}$$

As shown above, branching happens in twos, with one branch getting weight f_A and the other f_B. Similarly, the returns realized in each branch are modeled by infinite series of i.i.d. random variables A_i and B_i. The first branch will yield a return in the second period and the other branch in the third period, so we say that $t_A = 2$ and $t_B = 3$. In the interest of generality, consider the case in which the same behavior is exhibited for an arbitrary number of branches represented by the set $S = \{A, B, \dots\}$ with f_R and t_R for $R \in S$, and with each branch taking on the series of return values R_i in the course of resampling.

The growth rate I_a is such that, with $T = \max\{t_R : R \in S\}$,

$$I_a^T = \lim_{N\to\infty} \sqrt[N]{\left(\sum_{R\in S} R_1 f_R I_a^{T-t_R}\right) \dots \left(\sum_{R\in S} R_N f_R I_a^{T-t_R}\right)}$$

This is a natural extension of the conventional IRR, although it looks a bit different. Here, the recursive formulation of IRR is used. The formula is recursive in the sense that the reinvestment rate is defined in terms of the distribution of outcomes, which are themselves defined in terms of reinvestment rate. The final answer will be the fixed point of the expression.

Now we turn to computing I_a. Our first move is to drop into log-space:

$$I_a = \exp\left\{\frac{1}{T}\lim_{N\to\infty}\frac{1}{N}\left[\left(\log\sum_{R\in S} R_1 f_R I_a^{T-t_R}\right) + \dots + \left(\log\sum_{R\in S} R_N f_R I_a^{T-t_R}\right)\right]\right\}$$

Then, thanks to the Law of Large Numbers, the limit converges

$$I_a = \exp\left\{\frac{1}{T}\mathbb{E}\log\sum_{R\in S} R_1 f_R I_a^{T-t_R}\right\}$$

The actual value of I_a given random variables $R \in S$ of known distributions and fixed f_R and t_R can be solved for as the fixed point of the expression—that is, for $f(I)$ as follows:

$$f(I) = \exp\left\{\frac{1}{T}\mathbb{E}\log\sum_{R\in S} R_1 f_R I^{T-t_R}\right\}$$

You'll find that $f(f(f(\cdots f(1)\cdots))) \to I_a$.

This formulation of IRR has some nice properties. Specifically, when the values of the return variables are certain it reduces to conventional IRR. And when they are uncertain but all have the same periodicity, it reduces to the geometric mean. Since both are measures of growth rate, their synthesis goes a long way in validating the result presented here.